

Minimal Distance to Approximating Noncontextual System as a Measure of Contextuality

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Abstract

Let random vectors $R^c = \{R_p^c : p \in P_c\}$ represent joint measurements of certain subsets $P_c \subset P$ of *properties* $p \in P$ in different *contexts* $c \in C$. Such a system is traditionally called *noncontextual* if there exists a jointly distributed set $\{Q_p : p \in P\}$ of random variables such that R^c has the same distribution as $\{Q_p : p \in P_c\}$ for all $c \in C$. A trivial necessary condition for noncontextuality and a precondition for most approaches to measuring contextuality is that the system is *consistently connected*, i.e., all R_p^c, R_p^c, \dots measuring the same property $p \in P$ have the same distribution. The Contextuality-by-Default (CbD) approach allows detecting and measuring “true” contextuality on top of inconsistent connectedness, but at the price of a higher computational cost.

In this paper we propose a novel approach to measuring contextuality that shares the generality and basic definitions of the CbD approach and the computational benefits of the previously proposed Negative Probability (NP) approach. The present approach differs from CbD in that instead of considering all possible joints of the double-indexed random variables R_p^c , it considers all possible approximating *single-indexed* systems $\{Q_p : p \in P\}$. The degree of contextuality is defined based on the minimum possible probabilistic distance of the actual measurements R^c from $\{Q_p : p \in P_c\}$. We show that the defined measure agrees with a certain measure of contextuality of the CbD approach for all systems where each property enters in exactly two contexts. This measure can be calculated far more efficiently than the CbD measure and even more efficiently than the NP measure for sufficiently large systems. The present approach can be modified so as to agree with the NP measure of contextuality on all consistently connected systems while extending it to inconsistently connected systems.

1 Introduction

As a basic example, consider a so called Alice–Bob system with properties a_1, a_2 for Alice and b_1, b_2 for Bob measured in pairs (a_i, b_j) by the bivariate ± 1 -valued random variables (A_{ij}, B_{ij}) , $i, j \in \{1, 2\}$. Here i is called Alice’s setting and j is called Bob’s setting.

Denoting by $\langle X \rangle$ the expectation of a random variable X , suppose that over several trials with different settings (i, j) , one observes perfectly uniform marginals $\langle A_{ij} \rangle = \langle B_{ij} \rangle = 0$ for all $i, j \in \{1, 2\}$ together with the product expectations (correlations) $\langle A_{21}B_{21} \rangle = \langle A_{11}B_{11} \rangle = \langle A_{12}B_{12} \rangle = 1$ and $\langle A_{22}B_{22} \rangle = -1$ for the jointly measured pairs of properties. Assuming now that the system is *noncontextual* in the sense that it could be represented with fictitious single-indexed jointly distributed random variables (A_1, A_2, B_1, B_2) satisfying $(A_i, B_j) \sim (A_{ij}, B_{ij})$, where \sim stands for “has the same distribution as”, we can infer from the observed correlations that the value of A_2 is always equal to that of B_1 which is always equal to that of A_1 which is always equal to that of B_2 . However, the last correlation being -1 implies that the value of A_2 is always opposite to the measured value of B_2 . This contradiction implies that the system cannot be noncontextual — the single-indexed random variables representing Alice’s measurements cannot be jointly distributed with those representing Bob’s measurements. Several approaches have been developed for assessing such contextual properties of a system. In the rest of this section, we introduce these approaches specialized to the Alice–Bob system (the general case will be considered later in Section 3).

In the Contextuality-by-Default (CbD) approach [5, 6, 7, 8, 9, 10], the jointly distributed random variables (A_{ij}, B_{ij}) representing measurements in a given experimental context are called *bunches* and the sets of random variables $\{A_{i1}, A_{i2}\}$ and $\{B_{1j}, B_{2j}\}$ for $i, j \in \{1, 2\}$ representing measurements of the same property in different contexts are called *connections*. The random variables entering a connection are said to be *stochastically unrelated* since there is no pairing-scheme to obtain a joint distribution for the values observed in different experimental conditions. The system is said to be *consistently connected*¹ if $A_{i1} \sim A_{i2}$ and $B_{1j} \sim B_{2j}$ for all $i, j \in \{1, 2\}$, that is, if the distribution of A_{ij} measuring Alice’s property a_i does not depend on Bob’s setting j and the distribution of B_{ij} measuring Bob’s property b_j does not depend Alice’s setting i . If a system is not consistently connected, it is called *inconsistently connected*. An inconsistently connected system is contextual in a trivial way since there is a direct cross-influence from Bob’s setting to the distribution of Alice’s measurements or from Alice’s setting to the distribution of Bob’s measurements. However, the CbD approach allows detecting the presence of contextuality of the type of the opening example on top of inconsistent connectedness.

A *coupling* of random variables X_1, \dots, X_n (which may be stochastically unrelated) is any jointly distributed random vector $(\hat{X}_1, \dots, \hat{X}_n)$ that satisfies $\hat{X}_i \sim X_i$ for $i = 1, \dots, n$. The coupling $(\hat{X}_1, \dots, \hat{X}_n)$ is said to be *maximal*²

¹Consistent connectedness has also been referred to as non-signaling and marginal selectivity in other contexts.

²The maximal couplings $(\hat{X}_1, \dots, \hat{X}_n)$ of discrete random variables (X_1, \dots, X_n) satisfy

$$\Pr[\hat{X}_1 = \dots = \hat{X}_n = x] = \min_{i \in \{1, \dots, n\}} \Pr[X_i = x]$$

for all x (see Thorisson [15], Chapter 1, Theorem 4.2) and so the maximum coupling probability is given by the sum of the above probability over all x . The same result generalizes to densities of arbitrary random variables (see [15], Chapter 3, Section 7).

if the *coupling probability* $\Pr[\hat{X}_1 = \dots = \hat{X}_n]$ is maximum possible given the distributions of X_1, \dots, X_n . In the CbD approach, one considers couplings $\{(\hat{A}_{ij}, \hat{B}_{ij})\}_{i,j \in \{1,2\}}$ of the bunches $\{(A_{ij}, B_{ij})\}_{i,j \in \{1,2\}}$ as well as the subcouplings $(\hat{A}_{i1}, \hat{A}_{i2})$ and $(\hat{B}_{1j}, \hat{B}_{2j})$ for $i, j \in \{1, 2\}$ corresponding to connections. A system is said to be *noncontextual* if there exists a coupling $\{(\hat{A}_{ij}, \hat{B}_{ij})\}_{i,j \in \{1,2\}}$ of the bunches $\{(A_{ij}, B_{ij})\}_{i,j \in \{1,2\}}$ such that the subcouplings corresponding to connections are maximal.

For a consistently connected system, this definition reduces to the existence of a coupling $\{(\hat{A}_{ij}, \hat{B}_{ij})\}_{i,j \in \{1,2\}}$ of the bunches $\{(A_{ij}, B_{ij})\}_{i,j \in \{1,2\}}$ such that $\hat{A}_{i1} = \hat{A}_{i2}$ and $\hat{B}_{1j} = \hat{B}_{2j}$ for all $i, j \in \{1, 2\}$ (and is therefore equivalent to the traditional understanding of contextuality for consistently connected systems). If such a coupling cannot be found, one can measure how far from maximality the subcouplings are by determining the minimum possible value of

$$\Delta^{\text{CbD}} = \sum_{i \in \{1,2\}} \Pr[\hat{A}_{i1} \neq \hat{A}_{i2}] + \sum_{j \in \{1,2\}} \Pr[\hat{B}_{1j} \neq \hat{B}_{2j}] \quad (1)$$

over all couplings $\{(\hat{A}_{ij}, \hat{B}_{ij})\}_{i,j \in \{1,2\}}$ of the pairs $\{(A_{ij}, B_{ij})\}_{i,j \in \{1,2\}}$. When the system is inconsistently connected, the above measure of contextuality generalizes to the excess of Δ^{CbD} over its minimum possible value

$$\Delta_0^{\text{CbD}} = \sum_{i \in \{1,2\}} \frac{1}{2} |\langle A_{i1} \rangle - \langle A_{i2} \rangle| + \sum_{j \in \{1,2\}} \frac{1}{2} |\langle B_{1j} \rangle - \langle B_{2j} \rangle| \quad (2)$$

allowed by the marginal expectations $\langle A_{ij} \rangle, \langle B_{ij} \rangle, i, j \in \{1, 2\}$ so that the system is noncontextual in general if and only if $\Delta^{\text{CbD}} - \Delta_0^{\text{CbD}} = 0$. Other measures consistent with the definition of contextuality can be used with the CbD approach as well.

In the negative probability (NP) approach [1, 3, 13, 14], the system is said to be noncontextual if there exists a joint distribution of fictitious random variables (A_1, A_2, B_1, B_2) such that $(A_i, B_j) \sim (A_{ij}, B_{ij})$ for all $i, j \in \{1, 2\}$. If this is not possible, then the degree of contextuality is defined as the minimum possible negative probability mass

$$\begin{aligned} \Delta^{\text{NP}} &= \sum_{a_1, a_2, b_1, b_2 \in \{-1, 1\}} \min\{0, p(a_1, a_2, b_1, b_2)\} \\ &= -1 + \sum_{a_1, a_2, b_1, b_2 \in \{-1, 1\}} |p(a_1, a_2, b_1, b_2)|. \end{aligned}$$

of a negative probability joint $p(a_1, a_2, b_1, b_2)$ (a real-valued function summing to 1 over all $a_1, a_2, b_1, b_2 \in \{-1, 1\}$) of (A_1, A_2, B_1, B_2) such that $(A_i, B_j) \sim (A_{ij}, B_{ij})$ for all $i, j \in \{1, 2\}$, that is,

$$\sum_{a_3-i, b_3-j \in \{-1, 1\}} p(a_1, a_2, b_1, b_2) = \Pr[A_{ij} = a_i, B_{ij} = b_j]$$

for all $a_i, b_j \in \{-1, 1\}$ and $i, j \in \{1, 2\}$. Such a negative probability joint is known to exist provided that $\langle A_{i1} \rangle = \langle A_{i2} \rangle$ and $\langle B_{1j} \rangle = \langle B_{2j} \rangle$ for all $i, j \in \{1, 2\}$.

It has been shown [4] that both approaches give the same numerical values for the degree of contextuality for the simplest Alice–Bob system considered here as well as for some other simple systems. The CbD approach, however, is more general in that it can be directly applied even when $\langle A_{i1} \rangle \neq \langle A_{i2} \rangle$ or $\langle B_{1j} \rangle \neq \langle B_{2j} \rangle$ whereas the negative probability joint for single-indexed variables only exists when the system is consistently connected.

Although the CbD approach has wider applicability, the NP approach has the benefit of being computationally far more efficient. This is because the number of variables needed to represent the joint probabilities of a single-indexed joint is much smaller than the number of variables needed to represent the double-indexed coupling of the CbD approach.

In this paper, we present a novel approach to measuring contextuality that has the same generality and formal language as the CbD approach while in certain specializations reaching and potentially surpassing the computational efficiency of the NP approach. Thus, on these desiderata, the present approach appears to be superior to either of the previous approaches. Alternatively, the present approach can be formulated so as to agree with either approach although the computational benefits over the compared system will then be lost. In particular, a certain formulation agrees with the NP approach on all consistently connected systems while extending it to inconsistently connected systems.

In the following sections, we first sketch the ideas using the simplest Alice–Bob example considered above. Then, we formulate the general theory using mathematically rigorous definitions and theorems extending on the framework of the CbD approach. After that, we illustrate the computational benefits using the Alice–Bob example with m settings for Alice and n settings for Bob. We conclude by giving an abstract view to the logic of the present approach and compare its similarities and differences to CbD as well as consider their implications.

2 Defining an approximating non-contextual system: a sketch of the ideas

In this section, we introduce the basic ideas using the same Alice–Bob example as in the introduction.

Lemma 1. *Given jointly distributed ± 1 -valued random variables A , A' , and A'' , we have*

$$\frac{1}{2} |\langle A' \rangle - \langle A'' \rangle| \leq \Pr[A' \neq A''] \leq \Pr[A \neq A'] + \Pr[A \neq A'']. \quad (3)$$

Furthermore, for any given values of $\langle A' \rangle, \langle A'' \rangle \in [-1, 1]$ with $\langle A \rangle$ in between,

there exists jointly distributed (A, A', A'') such that both of the above inequalities hold as equality.

Proof. The maximum coupling probability of A' and A'' is given by

$$\begin{aligned} & \min \{ \Pr[A' = 1], \Pr[A'' = 1] \} + \min \{ \Pr[A' = -1], \Pr[A'' = -1] \} \\ &= \min \left\{ \frac{1}{2} + \frac{1}{2} \langle A' \rangle, \frac{1}{2} + \frac{1}{2} \langle A'' \rangle \right\} + \min \left\{ \frac{1}{2} - \frac{1}{2} \langle A' \rangle, \frac{1}{2} - \frac{1}{2} \langle A'' \rangle \right\} \\ &= 1 - \frac{1}{2} [\max \{ \langle A' \rangle, \langle A'' \rangle \} - \min \{ \langle A' \rangle, \langle A'' \rangle \}] = 1 - \frac{1}{2} |\langle A' \rangle - \langle A'' \rangle|, \end{aligned}$$

which yields the left inequality and the fact that it can hold as an equality for some joint with the given marginal expectations. The right inequality follows from the fact that the event $A' \neq A''$ is contained in the event $A \neq A'$ or $A \neq A''$ (since $A = A'$ and $A = A''$ imply $A' = A''$) and both inequalities can be made to hold as an equality by letting (A', A'') be a maximal coupling and then choose A to be either A' or A'' to make equality on the right. This yields the result for $\langle A \rangle$ equal to $\langle A' \rangle$ or $\langle A'' \rangle$; for a value of $\langle A \rangle$ between $\langle A' \rangle$ and $\langle A'' \rangle$, we can then take a convex combination of the joints of (A, A', A'') in the two cases. \square

Definition 2. We say that the Alice–Bob system is noncontextual, if there exists jointly distributed fictitious ± 1 -valued random variables (A_1, A_2, B_1, B_2) such that each pair (A_i, B_j) can be coupled with the observable pair (A_{ij}, B_{ij}) by a coupling

$$((\hat{A}_i, \hat{B}_j), (\hat{A}_{ij}, \hat{B}_{ij})) \quad (4)$$

such that over the four such couplings, the sum

$$\Delta = \sum_{i,j \in \{1,2\}} \left(\Pr[\hat{A}_{ij} \neq \hat{A}_i] + \Pr[\hat{B}_{ij} \neq \hat{B}_j] \right) \quad (5)$$

equals

$$\Delta_0 = \sum_{i \in \{1,2\}} \frac{1}{2} |\langle A_{i1} \rangle - \langle A_{i2} \rangle| + \sum_{j \in \{1,2\}} \frac{1}{2} |\langle B_{1j} \rangle - \langle B_{2j} \rangle|, \quad (6)$$

which is (by Lemma 1)³ the minimum possible value of (5) over all possible separate couplings $(\hat{A}_i, \hat{A}_{ij})$ and $(\hat{B}_j, \hat{B}_{ij})$ of respectively (A_i, A_{ij}) and (B_j, B_{ij}) for $i, j \in \{1, 2\}$ and all possible choices of jointly distributed ± 1 -valued random variables (A_1, A_2, B_1, B_2) . A measure of contextuality is given by the minimum

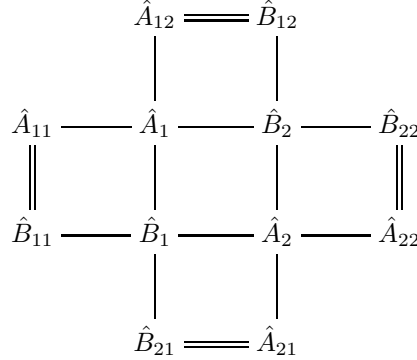
³The precise argument here is that for $i \in \{1, 2\}$ and any joint of $(\hat{A}_i, \hat{A}_{i1}, \hat{A}_{i2})$ having the marginal expectations $\langle A_i \rangle, \langle A_{i1} \rangle, \langle A_{i2} \rangle$, Lemma 1 implies that $\frac{1}{2} |\langle A_{i1} \rangle - \langle A_{i2} \rangle| \leq \Pr[\hat{A}_{i1} \neq \hat{A}_i] + \Pr[\hat{A}_{i2} \neq \hat{A}_i]$ with equality for some joint and analogously for $(\hat{B}_i, \hat{B}_{1j}, \hat{B}_{2j})$ for $j \in \{1, 2\}$. Given joints for which the equality holds, we obtain a Δ equaling Δ_0 with the couplings $(\hat{A}_i, \hat{A}_{ij})$ and $(\hat{B}_j, \hat{B}_{ij})$ defined as marginals of the 3-joints with (A_1, A_2, B_1, B_2) given as an arbitrary joint with the same marginals as $\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2$. Conversely, any A_1, A_2, B_1, B_2 together with couplings $(\hat{A}_i, \hat{A}_{ij})$ and $(\hat{B}_j, \hat{B}_{ij})$ for $i, j \in \{1, 2\}$ can be used to define joints for $(\hat{A}_i, \hat{A}_{i1}, \hat{A}_{i2})$ and $(\hat{B}_i, \hat{B}_{1j}, \hat{B}_{2j})$ whose 2-marginals agree with $(\hat{A}_i, \hat{A}_{ij})$ and $(\hat{B}_j, \hat{B}_{ij})$ and so it follows $\Delta \geq \Delta_0$ implying that no value smaller than Δ_0 can be obtained.

possible value of the difference $\Delta - \Delta_0$ over the possible couplings and choices of (A_1, A_2, B_1, B_2) .

Intuitively, the single-indexed system (A_1, A_2, B_1, B_2) in the above definition approximates the double-indexed system $\{(A_{ij}, B_{ij}) : i, j \in \{1, 2\}\}$ and the distance of the double-indexed system to the approximating single indexed-system is given by the minimum possible value of Δ over all possible couplings $((\hat{A}_i, \hat{B}_j), (\hat{A}_{ij}, \hat{B}_{ij}))$ of $((A_i, B_j), (A_{ij}, B_{ij}))$ for all $i, j \in \{1, 2\}$. If this distance equals Δ_0 , then the error of the approximation is the minimum possible allowed by the marginal distributions of A_{ij} and B_{ij} for $i, j \in \{1, 2\}$ and the system is considered noncontextual (apart from any inconsistent connectedness).

Theorem 3. *A system is noncontextual according to Definition 2 if and only if it is noncontextual according to the standard Cbd definition as well.*

Proof. The four couplings (4) together with the jointly distributed variables (A_1, A_2, B_1, B_2) induce a coupling $((\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2), \{(\hat{A}_{ij}, \hat{B}_{ij}) : i, j \in \{1, 2\}\})$ of all variables $((A_1, A_2, B_1, B_2), \{(A_{ij}, B_{ij}) : i, j \in \{1, 2\}\})$ with the pairs $(\hat{A}_{ij}, \hat{B}_{ij})$ independent of each other given $(\hat{A}_1, \hat{A}_2, \hat{B}_1, \hat{B}_2)$:



The subcoupling $\{(\hat{A}_{ij}, \hat{B}_{ij}) : i, j \in \{1, 2\}\}$ (leaving out the single indexed variables) is jointly distributed and (5) equaling (6) implies by Lemma 1 that

$$\begin{aligned} & \sum_{i \in \{1, 2\}} \Pr[\hat{A}_{i1} \neq \hat{A}_{i2}] + \sum_{j \in \{1, 2\}} \Pr[\hat{B}_{1j} \neq \hat{B}_{2j}] \\ &= \sum_{i \in \{1, 2\}} \frac{1}{2} \left| \langle \hat{A}_{i1} \rangle - \langle \hat{A}_{i2} \rangle \right| + \sum_{j \in \{1, 2\}} \frac{1}{2} \left| \langle \hat{B}_{1j} \rangle - \langle \hat{B}_{2j} \rangle \right|, \end{aligned} \quad (7)$$

which is precisely the criterion for noncontextuality according to the standard Cbd definition given by (1) and (2).

Conversely, suppose that (7) holds for some coupling $\{(\hat{A}_{ij}, \hat{B}_{ij}) : i, j \in \{1, 2\}\}$ of $\{(A_{ij}, B_{ij}) : i, j \in \{1, 2\}\}$. Then, for all $i, j \in \{1, 2\}$, we can choose \hat{A}_i and A_i both equal to \hat{A}_{ik} for arbitrary $k \in \{1, 2\}$ and \hat{B}_j and B_j both equal to \hat{B}_{kj} for arbitrary $k \in \{1, 2\}$. It immediately follows that $((\hat{A}_i, \hat{B}_j), (\hat{A}_{ij}, \hat{B}_{ij}))$

is a coupling of $((A_i, B_j), (A_{ij}, B_{ij}))$ for all $i, j \in \{1, 2\}$ and that these four couplings satisfy

$$\begin{aligned}
& \sum_{i,j \in \{1,2\}} \left(\Pr[\hat{A}_{ij} \neq \hat{A}_i] + \Pr[\hat{B}_{ij} \neq \hat{B}_j] \right) \\
&= \sum_{i \in \{1,2\}} \Pr[\hat{A}_{i1} \neq \hat{A}_{i2}] + \sum_{j \in \{1,2\}} \Pr[\hat{B}_{1j} \neq \hat{B}_{2j}] \\
&= \sum_{i \in \{1,2\}} \frac{1}{2} \left| \langle \hat{A}_{i1} \rangle - \langle \hat{A}_{i2} \rangle \right| + \sum_{j \in \{1,2\}} \frac{1}{2} \left| \langle \hat{B}_{1j} \rangle - \langle \hat{B}_{2j} \rangle \right|
\end{aligned}$$

so that Definition 2 is satisfied. \square

It is not essential for the idea of approximating one system with another that the approximating system is a proper noncontextual system. We can in fact define a concept of optimal approximation by any system that can predict consistently connected observable joint distributions:

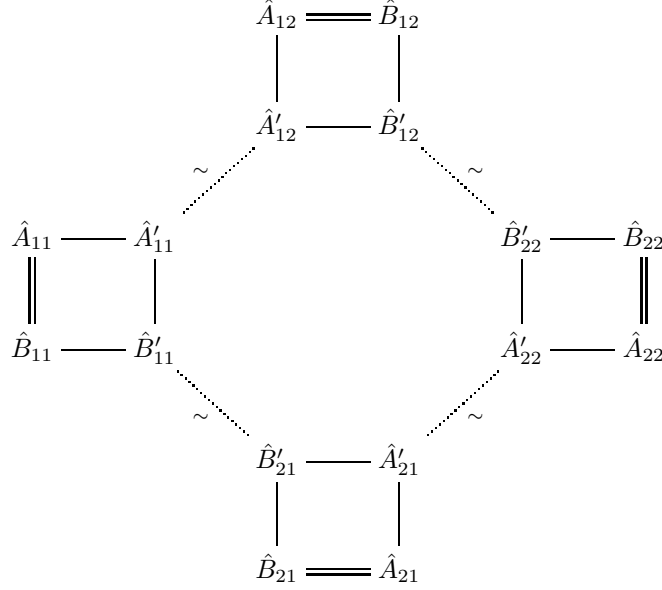
Definition 4. A set of bivariate random variables $\{(A'_{ij}, B'_{ij}) : i, j \in \{1, 2\}\}$ that is consistently connected, i.e., satisfies $A'_{ij} \sim A'_{ij'}$ and $B'_{ij} \sim B'_{i'j}$ for all $i, i', j, j' \in \{1, 2\}$, is said to *approximate optimally* the system $\{(A_{ij}, B_{ij}) : i, j \in \{1, 2\}\}$ if there exists couplings $((\hat{A}_{ij}, \hat{B}_{ij}), (\hat{A}'_{ij}, \hat{B}'_{ij}))$ for all $i, j \in \{1, 2\}$ such that

$$\Delta' = \sum_{i,j \in \{1,2\}} \left(\Pr[\hat{A}_{ij} \neq \hat{A}'_{ij}] + \Pr[\hat{B}_{ij} \neq \hat{B}'_{ij}] \right)$$

equals

$$\Delta_0 = \sum_{i \in \{1,2\}} \frac{1}{2} \left| \langle A_{i1} \rangle - \langle A_{i2} \rangle \right| + \sum_{j \in \{1,2\}} \frac{1}{2} \left| \langle B_{1j} \rangle - \langle B_{2j} \rangle \right|.$$

This construction is illustrated by the following diagram:



Remark 5. A system is noncontextual according to Definition 2 if and only if it is approximated optimally by a system that is noncontextual in the traditional sense.

Definition 4 above allows in particular approximating by a negative probability system:

Definition 6. Consider all negative probability joints of (A_1, A_2, B_1, B_2) having proper marginals for all (A_i, B_j) with $\Delta' = \Delta_0$ where we define $(A'_{ij}, B'_{ij}) = (A_i, B_j)$. The degree of contextuality in the system is then defined as the minimum possible total negative probability mass among all such optimally approximating negative probability joints.

Example 7. Definition 4 can also be applied to a specific model predicting a consistently connected set of jointly distributed pairs (A'_{ij}, B'_{ij}) , for example, to the quantum model

$$\langle A'_{ij} B'_{ij} \rangle = -\cos(\alpha_i - \beta_j), \quad \langle A'_{ij} \rangle = \langle B'_{ij} \rangle = 1/2, \quad i, j \in \{1, 2\}$$

of the Einstein–Podolsky–Rosen (EPR) experiment for photons (see, e.g., [2]). Thus, if the observations deviate somewhat from consistent connectedness, the above approach still allows one to test whether the observations are as close to the prediction as possible ignoring the contextual changes in the marginals. This allows, for example, $\langle A_{11} \rangle$ and $\langle A_{12} \rangle$ to deviate from the predicted value $1/2$ in some cases, but only if they deviate to different directions.

3 General definition and results

In this section, we formalize the ideas presented in the previous section into a general and rigorous mathematical formulation.

Let P denote the set of all properties and C the set of all contexts. For simplicity, we assume that there is a finite number of elements in C and P (although most of the results should be generalizable to countable C and P as well). Let $R^c = \{R_p^c : p \in P^c\}$ denote a set of jointly distributed random variables $R_p^c : \Omega_c \rightarrow E_p$ (with values in arbitrary spaces E_p) representing the observation of properties $p \in P^c$ that enter in context $c \in C$. Furthermore, let $C_p \subset C$ denote the set of all contexts in which the property p enters. Following the Contextuality-by-Default terminology, the jointly observable set R^c of random variables entering in a given context c will be called a *bunch* in the following (as will any analogously indexed set of jointly distributed random variables).

The CbD criterion of contextuality is given by the existence of a coupling $\{\hat{R}^c : c \in C\}$ of $\{R^c : c \in C\}$ satisfying $\Delta^{\text{CbD}} = \Delta_0^{\text{CbD}}$, where

$$\Delta^{\text{CbD}} = \sum_{p \in P} \Pr[\hat{R}_p^c \neq \hat{R}_p^{c'} \text{ for some } c, c' \in C_p] \quad (8)$$

and

$$\Delta_0^{\text{CbD}} = \sum_{p \in P} \min_{\substack{\text{all couplings} \\ \{T^c : c \in C_p\} \text{ of } \{R_p^c : c \in C_p\}}} \Pr[T^c \neq T^{c'} \text{ for some } c, c' \in C_p]. \quad (9)$$

As discussed in Section 1, this condition implies that all subcouplings $\{\hat{R}_p^c : c \in C_p\}$ corresponding to a connection $\{R_p^c : c \in C_p\}$, $p \in P$, are maximal. This formulation of the criterion also yields a measure of contextuality $\Delta^{\text{CbD}} - \Delta_0^{\text{CbD}}$ to which we will compare the measure of contextuality of the present approach. Note however, that the CbD approach allows for other measures of contextuality as well.

Let us then give the general definitions of the present approach. Given jointly distributed single-indexed random variables $\{Q_p : p \in P\}$, we define the bunches $Q^c = \{Q_p : p \in P^c\}$ with elements indexed as $Q_p^c = Q_p$.

Given couplings (\hat{R}^c, \hat{Q}^c) of (R^c, Q^c) for every $c \in C$, let us denote

$$\Delta = \sum_{p \in P} \sum_{c \in C_p} \Pr[\hat{Q}_p^c \neq \hat{R}_p^c] \quad (10)$$

and

$$\Delta_0 = \sum_{p \in P} \Delta_p, \quad (11)$$

where

$$\Delta_p = \min_{\text{all choices of } Q_p} \sum_{c \in C_p} \min_{\substack{\text{all couplings } (T_p, S_p) \\ \text{of } (Q_p, R_p^c)}} \Pr[T_p \neq S_p]. \quad (12)$$

Remark 8. Note that \hat{Q}_p^c in (10) is distinct for every c (even if the original variables are determined by $Q_p^c = Q_p$ for all c).

Definition 9. The system is *noncontextual* if there exists jointly distributed random variables $\{Q_p : p \in P\}$ and couplings (\hat{R}^c, \hat{Q}^c) of (R^c, Q^c) for every $c \in C$ such that $\Delta = \Delta_0$. Furthermore, we define the minimum possible value of the difference $\Delta - \Delta_0$ over all possible choices of the random variables $\{Q_p : p \in P\}$ and couplings $\{(\hat{R}^c, \hat{Q}^c) : c \in C\}$ as a measure of contextuality.

Remark 10. For any joint distribution of $\{Q_p : p \in P\}$ and any distributions of the couplings $\{(\hat{R}^c, \hat{Q}^c) : c \in C\}$, we can construct all the random variables appearing in these so that they are jointly distributed on a common probability space and satisfy $\hat{Q}_p^c = Q_p$ for all $c \in C$ with the bunches $\{\hat{R}^c : c \in C\}$ independent of each other given $\{Q_p : p \in P\}$.

Remark 11. For given Q_p , the couplings (T_p, S_p) of (Q_p, R_p^c) that minimize the sum in (12) are precisely those that are maximal. Thus, for given $\{Q^c : c \in C\}$, the couplings (\hat{R}^c, \hat{Q}^c) of (R^c, Q^c) that yield (10) equal to the sum over $p \in P$ of the sum in (12) are precisely those whose subcouplings $(\hat{R}_p^c, \hat{Q}_p^c)$ for all $p \in P^c$, $c \in C$, are maximal.

Definition 12. A set $\{Q^c : c \in C\}$ of random bunches $Q^c = \{Q_p^c : p \in P^c\}$ satisfying consistent connectedness condition (i.e., $Q_p^c \sim Q_p^{c'}$ for all $c, c' \in C_p$ and all $p \in P$) is said to *approximate optimally* the set $\{R^c : c \in C\}$ of random bunches $R^c = \{R_p^c : p \in P^c\}$ if there exist couplings (\hat{R}^c, \hat{Q}^c) of (R^c, Q^c) for every $c \in C$ such that $\Delta = \Delta_0$.

Definition 13. An alternative measure of contextuality is given by the minimum negative probability mass over all negative probability joints of $\{Q_p : p \in P\}$ whose marginals $\{Q_p : p \in P^c\}$ approximate optimally the observed bunches $\{R^c : c \in C\}$. Expanding the definitions, this means that

1. for every $c \in C$, the marginal distribution for the subset $Q^c = \{Q_p : p \in P^c\}$ has proper nonnegative probabilities, and
2. over all $c \in C$, there exists couplings (\hat{R}^c, \hat{Q}^c) of (R^c, Q^c) for every $c \in C$ satisfying $\Delta = \Delta_0$.

Next, we will list several simple theorems characterizing some of the properties of the above definitions.

Theorem 14. *For a consistently connected system, the measure of Definition 13 agrees with the NP measure of contextuality.*

Proof. For a consistently connected system we have $\Delta_0 = 0$ and $\Delta = \Delta_0$ then implies that $\hat{Q}_p^c = \hat{R}_p^c$ for all $p \in P^c$ and $c \in C$ and so constraint 2 of Definition 13 reduces to the requirement that $\{Q_p : p \in P^c\} \sim \{R_p^c : p \in P^c\}$ for all c . This is precisely the standard NP approach. \square

Theorem 15. For a system where every property $p \in P$ enters in exactly two contexts c_p and c'_p , the minimum possible value of Δ over all couplings $\{(\hat{R}^c, \hat{Q}^c)\}$ for all $c \in C$ is the same as the minimum possible value of

$$\Delta^{\text{CbD}} = \sum_{p \in P} \Pr[\hat{R}_p^{c_p} \neq \hat{R}_p^{c'_p}] \quad (13)$$

over all couplings $\{\hat{R}^c : c \in C\}$ of $\{R^c : c \in C\}$ and Δ_0 equals

$$\Delta_0^{\text{CbD}} = \sum_{p \in P} \min_{\substack{\text{all couplings} \\ (T^{c_p}, T^{c'_p}) \text{ of } \{R_p^{c_p}, R_p^{c'_p}\}}} \Pr[T^{c_p} \neq T^{c'_p}]$$

This implies that under these assumptions, the measure of contextuality $\Delta - \Delta_0$ is equivalent to the CbD definition given by (8) and (9).

Proof. Let $\{(\hat{R}^c, \{\hat{Q}_p^c : p \in P^c\})\}$ for $c \in C$ be the couplings with the minimum possible value of Δ . As per Remark 10, we can assume all the couplings to be jointly distributed with $\hat{Q}_p^{c_p} = \hat{Q}_p^{c'_p}$. Then, $\hat{Q}_p^{c_p} = \hat{R}_p^{c_p}$ and $\hat{Q}_p^{c'_p} = \hat{R}_p^{c'_p}$ imply $\hat{R}_p^{c_p} = \hat{R}_p^{c'_p}$, which yields (by the same argument as in the proof of Lemma 1)

$$\Pr[\hat{R}_p^{c_p} \neq \hat{R}_p^{c'_p}] \leq \Pr[\hat{Q}_p^{c_p} \neq \hat{R}_p^{c_p}] + \Pr[\hat{Q}_p^{c'_p} \neq \hat{R}_p^{c'_p}].$$

Thus, considering $\{\hat{R}^c : c \in C\}$ as the coupling of $\{R^c : c \in C\}$ of the CbD approach, it follows that (13) is less than or equal to Δ . Conversely, suppose that $\{\hat{R}^c : c \in C\}$ is a coupling of $\{R^c : c \in C\}$ with the minimum possible value of (13). Then, by defining $Q_p = \hat{R}_p^{c_p}$ and taking (\hat{Q}^c, \hat{R}^c) with $\hat{Q}_p^{c_p} = \hat{Q}_p^{c'_p} = Q_p$ as the couplings of (Q^c, R^c) , we obtain

$$\Pr[\hat{Q}_p^{c_p} \neq \hat{R}_p^{c_p}] + \Pr[\hat{Q}_p^{c'_p} \neq \hat{R}_p^{c'_p}] = \Pr[\hat{R}_p^{c_p} \neq \hat{R}_p^{c'_p}]$$

and so Δ equals (13). Since the minimum possible values of Δ and Δ^{CbD} over their respective sets of considered couplings are equal, it follows that Δ_0 and Δ_0^{CbD} are also equal since they both minimize the equal minimal value of Δ and Δ^{CbD} over all systems whose marginals agree with those of $\{R^c : c \in C\}$. \square

Theorem 16. For ± 1 -valued random variables, we have

$$\Delta_p = \min_{\text{all choices of } Q_p} \frac{1}{2} \sum_{c \in C_p} |\langle R_p^c \rangle - \langle Q_p \rangle|$$

and the minimizers are characterized by

$$\langle Q_p \rangle = \text{Median}\{\langle R_p^c \rangle : c \in C_p\}.$$

When the number of elements in C_p is even, the median means any value between the two middle values. If the property p enters in exactly two contexts, $C_p = \{c_p, c'_p\}$, then Δ_p simplifies into

$$\Delta_p = \frac{1}{2} |\langle R_p^{c_p} \rangle - \langle R_p^{c'_p} \rangle|.$$

Proof. Since the maximal coupling probability of two ± 1 -valued random variables is given by equality on the left side of (3) in Lemma 1, it can be seen that for ± 1 -valued random variables, (12) reduces to finding the best L^1 -approximator to the means of all $\{R_p^c : c \in C_p\}$ as shown in the statement of this theorem. This problem is solved by the median. \square

Theorem 17. *A system is noncontextual according to Definition 9 if and only if there exists jointly distributed random variables $\{Q_p : p \in P\}$ that are minimizers of the sum in (12) and have the property that for each $c \in C$, the system consisting of the two bunches $S^1 = Q^c$ and $S^2 = R^c$ (both having the same set of properties) is noncontextual in the CbD sense (or in the sense of Definition 9, since the two are equal for systems with every property entering in precisely two contexts).*

Furthermore, for given $\{Q_p : p \in P\}$, the minimum possible value of Δ over all couplings (\hat{R}^c, \hat{Q}^c) of (R^c, Q^c) equals the sum over $c \in C$ of the minimum possible values of Δ^{CbD} over all couplings (\hat{S}^1, \hat{S}^2) of the subsystem $\{S^1, S^2\} = \{Q^c, R^c\}$ with each Δ^{CbD} corresponding to the partial sum

$$\sum_{p \in C_p} \Pr[\hat{Q}_p^c \neq \hat{R}_p^c]$$

appearing in (10).

Proof. Follows from Remark 11 and the definitions. \square

Theorem 18. *For a system of ± 1 -valued random variables, if exactly two properties p_c and p'_c enter a given context $c \in C$, then the minimum possible value of the partial sum*

$$\sum_{p \in C_p} \Pr[\hat{Q}_p^c \neq \hat{R}_p^c]$$

appearing in (10) over all couplings (\hat{R}^c, \hat{Q}^c) of (R^c, Q^c) is given by

$$\frac{1}{2} \max \left\{ |\langle Q_{p_c}^c Q_{p'_c}^c \rangle - \langle R_{p_c}^c R_{p'_c}^c \rangle|, |\langle Q_{p_c}^c \rangle - \langle R_{p_c}^c \rangle| + |\langle Q_{p'_c}^c \rangle - \langle R_{p'_c}^c \rangle| \right\}.$$

Proof. According to Theorem 17, this problem is equivalent to the problem of minimizing Δ^{CbD} given by (8) for the system given by the bunches $S^1 = (Q_{p_c}^c, Q_{p'_c}^c)$ and $S^2 = (R_{p_c}^c, R_{p'_c}^c)$ both having the same two properties identified by their position in the bunches. This is a so called cyclic-2 system for which the solution is known to be the above expression (see Eq. (4) in [11]). \square

4 Computational complexity

In this section we illustrate the computational benefits of the present approach using as example Alice–Bob systems with m settings for Alice and n settings for Bob.

First, consider again the Alice–Bob example with two settings for both Alice and Bob. In the standard CbD approach, the coupling

$$((\hat{A}_{11}, \hat{B}_{11}), (\hat{A}_{12}, \hat{B}_{12}), (\hat{A}_{21}, \hat{B}_{21}), (\hat{A}_{22}, \hat{B}_{22}))$$

is represented by the $2^8 = 256$ nonnegative variables

$$p_{a_{11}b_{11}a_{12}b_{12}a_{21}b_{21}a_{22}b_{22}} = \Pr[\hat{A}_{ij} = a_{ij}, \hat{B}_{ij} = b_{ij} : i, j \in \{1, 2\}],$$

representing the joint probabilities. For each $i, j \in \{1, 2\}$, there are 4 linear equations constraining the distribution of $(\hat{A}_{ij}, \hat{B}_{ij})$ to that of the observed pair (A_{ij}, B_{ij}) . The expression (13) can be evaluated directly from the values of these 256 variables and so the degree of contextuality is obtained by linear programming, minimizing this expression given the constraints (and subtracting Δ_0^{CbD}).

In the NP approach, the negative probability joint of the hypothetical single-indexed random variables (A_1, A_2, B_1, B_2) is represented by the 16 signed variables

$$p_{a_1a_2b_1b_2} = \Pr[A_1 = a_1, A_2 = a_2, B_1 = b_1, B_2 = b_2]. \quad (14)$$

For each $i, j \in \{1, 2\}$ there are 4 linear equations constraining the marginal of (A_i, B_j) of (14) to the observed joint. To minimize the total negative probability mass, the signed variables (14) have to be represented by their negative and positive parts,

$$p_{a_1a_2b_1b_2} = p_{a_1a_2b_1b_2}^+ - p_{a_1a_2b_1b_2}^-, \quad p_{a_1a_2b_1b_2}^+, p_{a_1a_2b_1b_2}^- \geq 0.$$

The total negative probability mass is then obtained as the linear expression $\sum_{a_1, a_2, b_1, b_2 \in \{+1, -1\}} p_{a_1a_2b_1b_2}^-$ and this expression can be minimized using linear programming given the $2 \cdot 16 = 32$ nonnegative variables and 16 equation constraints.

In the present approach, the approximating noncontextual system is represented by the 16 nonnegative variables

$$p_{a_1a_2b_1b_2} = \Pr[A_1 = a_1, A_2 = a_2, B_1 = b_1, B_2 = b_2] \quad (15)$$

and for each $i, j \in \{1, 2\}$, the coupling $((\hat{A}_i, \hat{B}_j), (\hat{A}_{ij}, \hat{B}_{ij}))$ of $((A_i, B_j), (A_{ij}, B_{ij}))$ is represented by the 16 nonnegative variables

$$p_{a_{ij}b_{ij}a_i b_i}^{ij} = \Pr[\hat{A}_{ij} = a_{ij}, \hat{B}_{ij} = b_{ij}, \hat{A}_i = a_i, \hat{B}_i = b_i]. \quad (16)$$

For each $i, j \in \{1, 2\}$ there are 4 linear equations constraining the marginal of $(\hat{A}_{ij}, \hat{B}_{ij})$ of (16) to the observed joint and another 4 linear equations constraining the marginal of (\hat{A}_i, \hat{B}_j) in (16) to agree with the marginal (A_i, B_j) in (15). The degree of contextuality is obtained by minimizing (5) under these constraints. This formulation has a total of $16 + 4 \cdot 16 = 80$ variables with nonnegativity constraints compared to the 256 of the standard CbD approach

Table 1: Linear programming problem size for the Contextuality-by-Default, Negative Probabilities, and the present approach with m settings for Alice and n settings for Bob (± 1 outcomes). The linear programming problem is of the form $Mq = p$ subject to $q \geq 0$, where M is a matrix determined by m, n and p is a vector of the probabilities of the possible joint outcomes in each context (padded by the same number of 0's in the present approach).

| | CbD | NP | Present approach |
|-----------------------------------------|----------|-------------|------------------|
| Nonnegative variables (columns of M) | 2^{mn} | 2^{m+n+1} | $2^{m+n} + 16mn$ |
| Equation constraints (rows of M) | $4mn$ | $4mn$ | $8mn$ |
| Inequality constraints | 0 | 0 | 0 |

and it has $4 \cdot 4 + 4 \cdot 4 = 32$ linear equations compared to the $4 \cdot 4 = 16$ of the standard CbD approach.

For a more general Alice–Bob system, with m settings for Alice and n settings for Bob, the number of nonnegative variables needed for the present approach is $2^{m+n} + 16mn$ with $8mn$ equation constraints. For the standard CbD approach, there are 2^{mn} nonnegative variables with $4mn$ equation constraints and for the NP approach there are 2^{m+n+1} nonnegative variables with $4mn$ equation constraints. These numbers are summarized in Table 1. It is clear that based on the linear programming problem size, the present approach should be computationally more efficient than either the standard CbD or the NP approach: already for $m, n \geq 4$, the number of nonnegative variables needed in the linear programming task is smaller for the present approach than for either of the two previous approaches. The number of equation constraints needed is double in the present approach compared to either of the two previous approaches. However, this is in fact a benefit as well, since any nonredundant equation constraints are simply used to eliminate the same number of variables from the system by linear program solvers.

5 Conclusions

We conclude by considering the logic of the present approach in an abstract setting, with possible generalizations, and its relation to the previously proposed Contextuality-by-Default approach.

5.1 Approximation in abstract terms

Let us denote by \mathcal{S} the class of all systems, by $\mathcal{C} \subset \mathcal{S}$ the class of consistently connected systems, by $\mathcal{N} \subset \mathcal{C}$ the class of noncontextual systems in the traditional sense (i.e., determined by a single-indexed system of proper random variables), and by $\mathcal{S}_R \subset \mathcal{S}$ the class of systems $R' \in \mathcal{S}$ whose marginals agree with R (i.e., $R'_p{}^c \sim R_p^c$ for all $c \in C, p \in P_c$). In abstract terms, the main idea of the present approach is to model a (possibly) inconsistently connected observable system $R \in \mathcal{S}$ by a consistently connected system Q that approximates it

as closely as possible within a given subclass $\mathcal{C}_0 \subset \mathcal{C}$ of consistently connected systems. A certain computationally convenient metric⁴ $\Delta(R, Q)$ is defined to measure in a probabilistic sense how far the observed system R is from its closest approximation. If the minimum value of $\Delta(R, Q)$ over all $Q \in \mathcal{C}_0$ equals $\Delta_0(R)$, the minimum value of $\Delta(R', Q)$ over all $Q \in \mathcal{C}$ and all $R' \in \mathcal{S}_R$, then the approximation is said to be optimal.

If the approximation is optimal and the approximating system Q is a non-contextual system in the traditional sense (i.e., $Q \in \mathcal{N}$), then R is considered noncontextual (apart from the trivial cross-influences that cause the inconsistent connectedness). If optimal approximation by $Q \in \mathcal{N}$ is not possible, then contextuality can be measured either as the minimum possible value of $\Delta(R, Q) - \Delta_0(R)$ over all approximating $Q \in \mathcal{C}_0 = \mathcal{N}$ or one can consider all (contextual or not) optimally approximating $Q \in \mathcal{C}_0 = \mathcal{C}$ and apply and minimize any measure of contextuality on all such Q . Since Q is consistently connected, the latter formulation allows generalizing any measure of contextuality that is defined for consistently connected systems to inconsistently systems. As an example, we have done this for the previously defined negative probability (NP) measure of contextuality (see Theorem 14). Thus, a measure of contextuality can be considered on the “outside” as the minimal distance from an approximating noncontextual system or “inside”, by applying and minimizing an existing measure of contextuality over all optimally approximating consistently connected systems.

For now, we have defined the concept of optimal approximation so that the distance $\Delta(R, Q)$ can be positive only by the amount $\Delta_0(R)$ required by the inconsistency of the marginal distributions of R_p^c , $c \in C$, $p \in P_c$. However, if the considered class \mathcal{C}_0 of models does not include all possible consistently connected marginals, one possible generalization is to change the definition of the threshold $\Delta_0(R)$ so that $\Delta(R', Q)$ is minimized over $R' \in \mathcal{S}_R$ and $Q \in \mathcal{C}_0$ (as opposed to $R' \in \mathcal{S}_R$ and $Q \in \mathcal{C}$). This would amount to testing if there is contextuality on top of that predicted by the considered class \mathcal{C}_0 of approximating systems and on top of any changes of the marginals from those allowed by the class \mathcal{C}_0 of approximating systems. In Example 7, for example, \mathcal{C}_0 is defined by a single model with uniform marginals and so our standard definition would never consider an observed system noncontextual if all the observed marginal expectations of some property deviate from the predicted 0 to the same direction (this is because optimal approximation would in that case require the marginal expectations of Q to deviate to the same direction which is not allowed by the predicted uniform marginals of $Q \in \mathcal{C}_0$). However, $\Delta_0(R', Q)$ optimized over $R' \in \mathcal{S}_R$ and $Q \in \mathcal{C}_0$ would yield a threshold that allows detecting if the distance of the approximating system $Q \in \mathcal{C}_0$ is not larger than the minimum required by the changes in the marginals from those allowed by \mathcal{C}_0 .

⁴It is easy to show that Δ given by (10) and Definition 12 for couplings (\hat{R}^c, \hat{Q}^c) of (R^c, Q^c) that minimize (10) is indeed a metric when taken as a function of the two sets of bunches $R = \{R^c : c \in C\}$ and $Q = \{Q^c : c \in C\}$.

5.2 Relation to Contextuality-by-Default

The main difference of the present approach to CbD is that the present approach does not consider a coupling of *all* random variables $\{R_p^c : c \in C, p \in P\}$ but only refers to couplings of one bunch (with the corresponding bunch of the approximating system) at a time in the definition of Δ (and subcouplings of those in the definition of Δ_0). As shown by proofs analogous to those used above, given the coupling $\{\hat{R}^c : c \in C\}$ of $\{R^c : c \in C\}$ of the CbD approach, one can always make an arbitrary choice of $c_p \in C_p$ for all $p \in P$ to define a single-indexed system $Q_p = \hat{R}_p^{c_p}$, $p \in P$, that approximates in some sense the observed system. For systems with each property entering in precisely two contexts, this single-indexed system approximates optimally (in the sense of the present approach) the observed system if and only if the subcouplings of $\{\hat{R}^c : c \in C\}$ corresponding to connections are maximal, that is, if the system is noncontextual in the CbD sense (see Theorem 15). This allows in particular all the analytic results of the so-called cyclic systems [12, 11, 9] derived in the CbD approach to be used for the present approach. Conversely, due to the equivalence, the computational advantages of the present system can be made use of to replace the calculations of the CbD approach in all systems with each property entering in precisely two contexts.

However, the present approach is not consistent with CbD in general. That is, there are systems that are contextual according to one definition but not according to the other. Still, the general idea of approximating optimally an inconsistently connected system with a consistently connected one can be formulated in a way that is consistent with CbD. For completeness, we will outline such a formulation here.

Given a coupling $\{\hat{R}^c : c \in C\} \cup \{\hat{Q}_p : p \in P\}$ of the bunches $\{R^c : c \in C\}$ and jointly distributed random variables $\{Q_p : p \in P\}$, denote

$$\Delta^* = \sum_{p \in P} \Pr[\hat{Q}_p \neq \hat{R}_p^c \text{ for some } c \in C_p] \quad (17)$$

and

$$\Delta_0^* = \sum_{p \in P} \Delta_p^*, \quad (18)$$

where

$$\Delta_p^* = \min_{\text{all choices of } Q_p} \min_{\substack{\text{all couplings } (T, \{S^c : c \in C_p\}) \\ \text{of } (Q_p, \{R_p^c : c \in C_p\})}} \Pr[T \neq S^c \text{ for some } c \in C_p]. \quad (19)$$

A system is considered contextual if $\Delta^* = \Delta_0^*$ and a measure of contextuality is given by $\Delta^* - \Delta_0^*$.

Using analogous proofs as above, it can be shown that this condition for contextuality is equivalent to that of the CbD approach and that the resulting measure of contextuality is equivalent to the one we have used for the CbD approach. However, being compatible with the CbD approach means that we have to consider a coupling of *all* variables and so the metric Δ^* is not a function

of the bunches of the original and approximating system but its arguments are full couplings of all random variables of a system and additionally the two couplings given as arguments must be defined on the same sample space. This means in particular that the computational benefits resulting from the simpler, single-indexed representation will be lost.

Conversely, one can also modify the CbD approach so as to be equivalent with the standard definition of the present approach. This would amount to changing the requirement of the maximality of each subcoupling $\{\hat{R}_p^c : c \in C_p\}$ corresponding to a connection $\{R_p^c : c \in C_p\}$, $p \in P$, to requiring that there exists a random variable Q_p satisfying

$$\sum_{c \in C_p} \Pr[Q_p \neq \hat{R}_p^c] = \Delta_p.$$

Then, the random variables $\{Q_p : p \in P\}$ would define an optimally approximating system satisfying the present definition of noncontextuality.

The remaining question is whether the definition of contextuality of the CbD approach or the definition of the present approach is more useful or if one may be more useful in some contexts and the other in some other contexts. An obvious example of difference between the two definitions is given when for each property $p \in P$, the marginal distributions of its observations in two different contexts have disjoint supports. In that case, the system is always noncontextual according to the CbD definition, since all subcouplings corresponding to the connections are maximal. However, according to the present approach, such a system may be contextual. We will construct one such example.

Suppose the system consists of four bunches R^1, \dots, R^4 , each measuring the same two ± 1 -valued properties $p \in \{1, 2\}$. Suppose $R_1^1 = R_2^1$ and $R_1^2 = -R_2^2$ are both uniformly distributed (so that $\langle R_1^1 \rangle = \langle R_2^1 \rangle = \langle R_1^2 \rangle = \langle R_2^2 \rangle = 0$, $\langle R_1^1 R_2^1 \rangle = 1$, and $\langle R_1^2 R_2^2 \rangle = -1$) and $R_1^3 = R_2^3 = 1$ and $R_1^4 = R_2^4 = -1$ (so that $\langle R_1^3 \rangle = \langle R_2^3 \rangle = 1$, $\langle R_1^4 \rangle = \langle R_2^4 \rangle = -1$, and $\langle R_1^3 R_2^3 \rangle = \langle R_1^4 R_2^4 \rangle = 1$). Then, R_p^3 and R_p^4 have disjoint supports for $p \in \{1, 2\}$ and so the system is noncontextual in the CbD sense. Also, the median of $\{\langle R_p^c \rangle : c \in \{1, 2, 3, 4\}\} = \{0, 0, 1, -1\}$ is 0 for $p \in \{1, 2\}$ and so, according to Theorem 16, $\langle Q_p \rangle = 0$ for $p \in \{1, 2\}$ is a necessary condition for $\Delta = \Delta_0$. Theorem 16 also yields the value $\Delta_0 = 2$ from the differences of marginals: $\Delta_p = \frac{1}{2}(|0 - 0| + |0 - 0| + |1 - 0| + |-1 - 0|) = 1$ for $p = 1, 2$. Now, according to Theorem 18 the minimum possible value of the partial sum

$$\sum_{p \in \{1, 2\}} \Pr[\hat{Q}_p^c \neq \hat{R}_p^c]$$

appearing in the definition of Δ is given by

$$\frac{1}{2} \max \{ |\langle Q_1 Q_2 \rangle - \langle R_1^c R_2^c \rangle|, |\langle Q_1 \rangle - \langle R_1^c \rangle| + |\langle Q_2 \rangle - \langle R_2^c \rangle| \}.$$

The sum of this expression over $c \in \{1, 2, 3, 4\}$ is

$$\begin{aligned}
& \frac{1}{2} \max \{|\rho - 1|, |0 - 0| + |0 - 0|\} \\
& + \frac{1}{2} \max \{|\rho - (-1)|, |0 - 0| + |0 - 0|\} \\
& + \frac{1}{2} \max \{|\rho - 1|, |0 - 1| + |0 - 1|\} \\
& + \frac{1}{2} \max \{|\rho - 1|, |0 - (-1)| + |0 - (-1)|\} \\
& = \frac{1}{2}(1 - \rho) + \frac{1}{2}(1 + \rho) + 1 + 1 = 3 > 2 = \Delta_0
\end{aligned}$$

for all $\rho \in [-1, 1]$ and so the system is contextual according to the present approach.

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